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Abstract

Let $M_n \subset \mathbb{R}^{n+1}$ be a complete hyperbolic affine hypersphere with mean curvature H, $H < 0$, and let C be its cubic form. We derive a differential inequality and an upper bound on the scalar function $||C||_\infty$ defined by the fiber-wise maximum of the value of C on the unit sphere bundle of M. The bound is attained for the affine hyperspheres which are asymptotic to a simplicial cone.

1 Introduction

The subject of this paper are complete hyperbolic affine hyperspheres. The Calabi conjecture [2] states that every hyperbolic affine hypersphere M which is complete (in the affine metric or in a metric induced by some Euclidean metric of the ambient affine space) is asymptotic to a regular (with nonempty interior and containing no lines) convex cone in the ambient space, with vertex in the center of M, and conversely, for every regular convex cone K in the ambient space and every negative real number H there exists a unique hyperbolic affine hypersphere with center in the vertex of K and with mean curvature H which is asymptotic to K. The conjecture has been proven by the efforts of many authors, a synthesis of the proof is given in [5, Section 2].

In [2, Lemma 5.2, p.31] Calabi obtained a differential inequality on the maximal eigenvalue of the Ricci tensor on complete hyperbolic affine hyperspheres and proved that the Ricci curvature has to be non-positive [2, Theorem 5.1, p.31]. He also showed that this bound is sharp, by presenting an example of a complete hyperbolic affine hypersphere with flat affine metric, namely the one which is asymptotic to a simplicial cone [2, p.37]. Similar differential inequalities and bounds can be obtained for the Pick invariant [9, eq. (2.5)], [5, Cor. 2.6.5, p.128].

In this contribution we use the techniques of [2] to obtain a differential inequality (Theorem 3.1) and a bound (Corollary 4.1) on the function

$$\Upsilon(x) = \max_{\xi \in T_x M, ||\xi||=1} (C(x))(\xi, \xi, \xi) \tag{1}$$

on a complete hyperbolic affine hypersphere M, with C being the cubic form. This function can be considered as the (point-wise) ∞-norm of the cubic form. The main motivation for studying this function lies in conic optimization, to which a link can be made as follows. An n-dimensional complete hyperbolic affine hypersphere M with center in the origin of $\mathbb{R}^{n+1}$ is asymptotic to some regular convex cone $K \subset \mathbb{R}^{n+1}$. On the interior of K define a logarithmically homogeneous convex function F by the relation $F[\alpha M] = \{-\log \alpha\}$ for all $\alpha > 0$ (cf. [6]). Then a bound on (1) translates into a bound on the self-concordance parameter [7, Sect. 2.3.3] of an appropriate multiple of F. We will elaborate on this relation in a subsequent publication (cf. also [4, Theorem 4.8]). Our bound on the cubic form is sharp, as we will demonstrate on the example of the affine hypersphere asymptotic to a simplicial cone (Proposition 5.1).

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2 Notations and preliminaries

Let us recall the definition of the cubic form and the expression for its Laplacian on affine hyperspheres. Let M be an n-dimensional complete hyperbolic affine hypersphere with mean curvature $H < 0$, g the affine metric, ∇ the Levi-Civita connection of the affine metric, and $\bar{\nabla}$ the affine connection induced by the ambient real affine space. The *cubic form* is defined as the covariant derivative $C = \bar{\nabla} g$ [8, eq. (2.6), p.34; Theorem 3.3, p.42]. The *difference tensor*[1] is defined as the difference $\bar{\nabla} - \nabla$ and equals $A = -\frac{1}{2}C$ [8, Prop.4.1, p.50]. The difference tensor and hence the cubic form are symmetric in all three indices [2, pp.23-24]. The Ricci curvature of the affine metric and the Laplacian of C with respect to the affine metric are given by [9, p.3]

$$R_{ij} = (n-1)Hg_{ij} + \frac{1}{4}C_i^{rs}C_{jrs}, \tag{2}$$

$$\Delta C_{ijk} = (n+1)HC_{ijk} + \frac{1}{4}C^{abc}(C_{abi}C_{cjk} + C_{abj}C_{cki} + C_{abk}C_{cij}) - \frac{1}{2}C_{ib}^a C_{jc}^b C_{ka}^c. \tag{3}$$

3 Differential inequality

Our purpose is to derive a differential inequality on the function Υ defined by (1) on M. Let $p \in M$ be an arbitrary point and let $\xi \in T_pM$ be a maximizer of the cubic form on the unit sphere in T_pM, $\Upsilon(p) = (C(p))(\xi,\xi,\xi)$, $||\xi|| = 1$.

Transport ξ from p to a neighbourhood U of p along the geodesics through p by means of the Levi-Civita connection to obtain a smooth unit length vector field on U. Then ξ satisfies

$$\nabla\xi = 0, \qquad \Delta\xi = 0 \tag{4}$$

at p [2, p.32], and $\bar{\Upsilon} = C_{ijk}\xi^i\xi^j\xi^k \leq \Upsilon$ on U, with equality attained at p. Define the symmetric second-order tensor $B = \bar{\nabla}_\xi g$ on U, in index form $B_{ij} = C_{ijk}\xi^k$. Note that B is traceless by the apolarity condition $C_{ij}^i = 0$ [8, Theorem 4.3, p.51], and that $B_{ij}\xi^i\xi^j = \bar{\Upsilon}$. Moreover, by [5, Lemma 2.2.3.19, p.106] we have at p that $B_{ij}\xi^i\eta^j = 0$ and $B_{ij}\eta^i\eta^j \leq \frac{1}{2}B_{ij}\xi^i\xi^j \leq \bar{\Upsilon}$ for every unit length vector η which is orthogonal to ξ. It follows that ξ is also a maximizer of $B(p)$ on the unit sphere in T_pM. In particular, at p we have

$$B_{ij}\xi^j = \bar{\Upsilon}g_{ij}\xi^j \tag{5}$$

as the first order optimality condition, and $\bar{\Upsilon}$ is the maximal eigenvalue of the matrix of $B(p)$ in any orthonormal basis of T_pM.

Let us now estimate the Laplacian of $\bar{\Upsilon}$. At p we have by virtue of (3),(4), and (5)

$$\begin{aligned} \Delta\bar{\Upsilon} &= (\Delta C)_{ijk}\xi^i\xi^j\xi^k = (n+1)H\bar{\Upsilon} + \frac{3}{4}C^{abc}C_{abi}C_{cjk}\xi^i\xi^j\xi^k - \frac{1}{2}C_{ib}^a C_{jc}^b C_{ka}^c \xi^i\xi^j\xi^k \\ &= (n+1)H\bar{\Upsilon} + \frac{3}{4}C^{abc}B_{ab}B_{cj}\xi^j - \frac{1}{2}B_b^a B_c^b B_a^c = (n+1)H\bar{\Upsilon} + \frac{3}{4}\bar{\Upsilon}B^{ab}B_{ab} - \frac{1}{2}B_b^a B_c^b B_a^c \\ &\geq (n+1)H\bar{\Upsilon} + \frac{n(n+1)}{4(n-1)^2}\bar{\Upsilon}^3. \end{aligned}$$

Here the inequality follows from Corollary A.2 in the Appendix. We obtain the following result.

Theorem 3.1. *Let M be an n-dimensional complete hyperbolic affine hypersphere with mean curvature $H < 0$, and let C be its cubic form. Then the function $\Upsilon(x) = \max_{\xi \in T_xM, ||\xi||=1}(C(x))(\xi,\xi,\xi)$ satisfies the differential inequality $\Delta\Upsilon \geq (n+1)H\Upsilon + \frac{n(n+1)}{4(n-1)^2}\Upsilon^3$ weakly in the sense of [1, Def. 1, p.46].* □

[1] Sometimes, such as in [2], the difference tensor is called cubic form, which can lead to serious confusions and apparently resulted in a missing factor of 2 in [3, Cor. 2, p.857] for the bound on the Pick invariant.

4 Bound

From Theorem 3.1 we might obtain a bound on the function (1) on a complete hyperbolic affine hypersphere. Namely, from [3, Cor. 1, p.857][2] it follows that $\Upsilon \leq \sqrt{-\frac{4H(n-1)^2}{n}} = \frac{2(n-1)\sqrt{-H}}{\sqrt{n}}$. As in the case of the Pick invariant [5, Cor. 2.6.5, p.128], however, there exists a purely algebraic proof.

As in the preceding section, let $p \in M$ be an arbitrary point, let $\xi \in T_pM$ be a maximizer of the cubic form on the unit sphere in T_pM, and define the traceless symmetric second-order tensor $B = \bar{\nabla}_\xi C$ on T_pM. Let $\{\xi, \eta_1, \dots, \eta_{n-1}\}$ be an orthonormal basis of T_pM, then we have

$$\begin{aligned}
\Upsilon^2(p) &= (B_{ij}\xi^i\xi^j)^2 = \left(-\sum_{k=1}^{n-1} B_{ij}\eta_k^i\eta_k^j\right)^2 \leq (n-1)\sum_{k=1}^{n-1}(B_{ij}\eta_k^i\eta_k^j)^2 \leq (n-1)\sum_{k=1}^{n-1} B_{il}B_j^l\eta_k^i\eta_k^j \\
&= (n-1)\left(B_{il}B^{li} - B_{il}B_j^l\xi^i\xi^j\right) = (n-1)\left(C_{ilj}C_k^{li}\xi^j\xi^k - \Upsilon^2(p)\right) \\
&= 4(n-1)(R_{jk} - (n-1)Hg_{jk})\xi^j\xi^k - (n-1)\Upsilon^2(p) \leq -4(n-1)^2H - (n-1)\Upsilon^2(p).
\end{aligned}$$

Here we used (5) and (2) in the last two equalities, respectively, and the non-positivity of the Ricci curvature [2, Theorem 5.1, p.31] in the last inequality. We obtain the following result.

Corollary 4.1. *Let M be an n-dimensional complete hyperbolic affine hypersphere with mean curvature $H < 0$, and let C be its cubic form. Then the function $\Upsilon(x) = \max_{\xi \in T_xM, ||\xi||=1}(C(x))(\xi,\xi,\xi)$ satisfies the inequality $\Upsilon \leq \frac{2(n-1)\sqrt{-H}}{\sqrt{n}}$.* □

5 Affine hyperspheres asymptotic to a simplicial cone

In this section we show that the inequalities in Theorem 3.1 and Corollary 4.1 are saturated for the affine hyperspheres which are asymptotic to a simplicial cone.

Denote by I_k the $k \times k$ identity matrix, by $\mathbf{1}_k \in \mathbb{R}^k$ the all-ones vector, and by $e_k \in \mathbb{R}^n$ the k-th canonical basis vector. Let $K = \mathbb{R}_+^{n+1}$ be the nonnegative orthant, $H < 0$ a negative real number, and let $M \subset K$ be the hyperbolic affine hypersphere with mean curvature H which is asymptotic to K. It is well-known that M is given by the equation $x_0 \cdot \dots \cdot x_n = c$ for some $c > 0$. On the interior of K we introduce the coordinates $y_0, \dots, y_n$ by

$$\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \gamma U \log x = \gamma \begin{pmatrix} \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{n+1}}\mathbf{1}_n^T \\ -\frac{1}{\sqrt{n+1}}\mathbf{1}_n & I_n + \frac{1}{n}\left(\frac{1}{\sqrt{n+1}} - 1\right)\mathbf{1}_n\mathbf{1}_n^T \end{pmatrix} \begin{pmatrix} \log x_0 \\ \vdots \\ \log x_n \end{pmatrix},$$

where $\gamma > 0$ is a constant to be determined later. Then in the y-coordinates the surface M is given by the equation $y_0 = \frac{\gamma \log c}{\sqrt{n+1}}$. The remaining coordinates $y_1, \dots, y_n \in \mathbb{R}$ define a global coordinate chart on M.

It is easily seen that the subgroup of unimodular diagonal automorphisms of K, which leaves M invariant, acts by translations on M. The affine metric g and the cubic form C of M have thus constant coefficients. The symmetric group S_{n+1}, which acts by permutations of the coordinates x_i on K, has a unique fixed point $y^0 \in M$ given by $y_1 = \dots = y_n = 0$. Therefore S_{n+1} acts on the tangent space $T_{y^0}M$, and both g and C have to be invariant under this action. It is not hard to check that the subgroup $S_n \subset S_{n+1}$ permuting the coordinates $x_1, \dots, x_n$ on K acts just by a corresponding permutation of the

[2]Obviously, in the formulation of this Corollary it should read $d_1 > 0$, $d_2 < 0$, and $u \leq \sqrt{-d_2/d_1}$, otherwise the assertion of the Corollary can be disproved simply by choosing u to be an appropriate constant. The proof of the Corollary contains several flaws, e.g., it assumes that a lower bound on the Ricci curvature implies an upper bound on the quantity $r\Delta r$, where r is the distance function from some point p [3, p.856]. Consider the unit ball in $\mathbb{R}^n$ equipped with the hyperbolic geometry given by the metric $g(x) = \frac{(1-R^2)I + xx^T}{(1-R^2)^2}$ in matrix form, where $R = \sqrt{x^Tx}$ is the Euclidean distance from the centre and I the identity matrix. The geodesic distance function from the centre is then given by $r = \frac{1}{2}\log\frac{1+R}{1-R}$, and $\Delta r = \frac{n-1}{R}$. Hence $r\Delta r$ tends to $+\infty$ as $R \to 1$. It is, however, straightforward to write down a correct proof of the Corollary, e.g., by setting $f = (a^2 - r^2)u$ [3, p.856]. With the choice $f = (1 + \cos\frac{\pi r^2}{a^2})u$ one can handle also differential inequalities of the form $\Delta\psi \geq d_1u^2 + d_2u$.

y-coordinates. Therefore the coefficients g_{ij}, C_{ijk} depend only on the number of distinct indices in the sets $\{i,j\}$ and $\{i,j,k\}$, respectively. Denote the latter number by θ_{ijk}. The permutation $\sigma_i \in S_{n+1}$ exchanging the coordinates x_0, x_i acts on $T_{y^0}M$ by the orthogonal transformation

$$U_i = I - \left(\frac{1-\sqrt{n+1}}{n}\mathbf{1}_n - e_i\right)\left(\frac{1-\sqrt{n+1}}{n}\mathbf{1}_n - e_i\right)^T.$$

It is not difficult to check that the invariance of g and C with respect to the transformations given by U_i determines these tensors up to a constant multiple. Namely, the matrix of the metric g has to be proportional to the identity matrix, and the cubic form C is given by

$$C_{ijk} = \begin{cases} -\alpha(n-1)\left(2+(n-2)\sqrt{n+1}\right), & \theta_{ijk}=1, \\ \alpha\left(2+(n-2)\sqrt{n+1}\right), & \theta_{ijk}=2, \\ \alpha\left(n+2-2\sqrt{n+1}\right), & \theta_{ijk}=3, \end{cases} \tag{6}$$

where α is a yet to be determined proportionality constant.

Let us choose γ such that the matrix of g is given by I_n. Contracting (2) with the metric and taking into account that M is flat, we obtain $C^{irs}C_{irs} = -4n(n-1)H$. Inserting the values for C_{irs} from (6), we get $\alpha^2 n^5(n-1) = -4n(n-1)H$, yielding $\alpha = \pm\frac{2\sqrt{-H}}{n^2}$ (it can be checked that the sign is positive). Hence on the unit length vector $\frac{1}{\sqrt{n}}\mathbf{1}_n$ the cubic form has the value $\frac{\alpha n^3(n-1)}{n^{3/2}} = \pm\frac{2(n-1)\sqrt{-H}}{\sqrt{n}}$, finally giving $\Upsilon \geq \frac{2(n-1)\sqrt{-H}}{\sqrt{n}}$. From Corollary 4.1 we can then conclude that $\Upsilon \equiv \frac{2(n-1)\sqrt{-H}}{\sqrt{n}}$ on M. By the affine equivalence of an arbitrary simplicial cone to the nonnegative orthant we obtain the following result.

Proposition 5.1. *Let $M \subset \mathbb{R}^{n+1}$ be a complete hyperbolic affine hypersphere which is asymptotic to a simplicial cone. Then both the bound in Theorem 3.1 and in Corollary 4.1 are saturated.* □

In particular, this will allow us to show that the optimal self-concordance parameter of an arbitrary regular convex cone is not worse than the parameter of the standard barrier for the nonnegative orthant of the same dimension.

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A Matrix lemma

In this section we provide an auxiliary inequality.

Lemma A.1. *Let $n \geq 2$ and $\lambda_1, \ldots, \lambda_{n-1} \leq 1$ be such that $\sum_{i=1}^{n-1} \lambda_i = -1$. Then $\frac{3}{4}\sum_{i=1}^{n-1} \lambda_i^2 - \frac{1}{2}\sum_{i=1}^{n-1} \lambda_i^3 \geq \frac{3n-1}{4(n-1)^2}$.*

Proof. Define $c_i = (n-1)\lambda_i + 1$. Then $\sum_{i=1}^{n-1} c_i = 0$ and $c_i \leq n$ for all i. It follows that $c_i^3 \leq nc_i^2$ for all i. We then have

$$\begin{aligned}
\frac{3}{4}\sum_{i=1}^{n-1}\lambda_i^2 - \frac{1}{2}\sum_{i=1}^{n-1}\lambda^3 - \frac{3n-1}{4(n-1)^2} &= \frac{3}{4(n-1)^2}\sum_{i=1}^{n-1}(c_i-1)^2 - \frac{1}{2(n-1)^3}\sum_{i=1}^{n-1}(c_i-1)^3 - \frac{3n-1}{4(n-1)^2} \\
= \ & \left(\frac{3}{4(n-1)} + \frac{1}{2(n-1)^2} - \frac{3n-1}{4(n-1)^2}\right) - \left(\frac{3}{2(n-1)^2} + \frac{3}{2(n-1)^3}\right)\sum_{i=1}^{n-1} c_i \\
& + \left(\frac{3}{4(n-1)^2} + \frac{3}{2(n-1)^3}\right)\sum_{i=1}^{n-1} c_i^2 - \frac{1}{2(n-1)^3}\sum_{i=1}^{n-1} c_i^3 \\
\geq \ & \left(\frac{3(n+1)}{4(n-1)^3} - \frac{n}{2(n-1)^3}\right)\sum_{i=1}^{n-1} c_i^2 = \frac{n+3}{4(n-1)^3}\sum_{i=1}^{n-1} c_i^2 \geq 0.
\end{aligned}$$

This completes the proof. □

Corollary A.2. *Let B be a real symmetric $n \times n$ matrix with vanishing trace. Then $\frac{3}{4}\lambda_{\max}(B)\operatorname{tr} B^2 - \frac{1}{2}\operatorname{tr} B^3 \geq \frac{n(n+1)}{4(n-1)^2}\lambda_{\max}^3(B)$.*

Proof. If $\lambda_{\max}(B) = 0$, then $B = 0$ and the assertion of the corollary is evident.

Suppose now that $\lambda_{\max}(B) > 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = 1$ be the eigenvalues of the matrix $\tilde{B} = \frac{B}{\lambda_{\max}(B)}$ in increasing order. Then $\sum_{i=1}^{n-1} \lambda_i = -1$ and $\lambda_i \leq 1$ for all i. We then have

$$\begin{aligned}
\frac{3}{4}\lambda_{\max}(B)\operatorname{tr} B^2 - \frac{1}{2}\operatorname{tr} B^3 - \frac{n(n+1)}{4(n-1)^2}\lambda_{\max}^3(B) &= \lambda_{\max}^3(B)\left(\frac{3}{4}\operatorname{tr}\tilde{B}^2 - \frac{1}{2}\operatorname{tr}\tilde{B}^3 - \frac{n(n+1)}{4(n-1)^2}\right) \\
= \ \lambda_{\max}^3(B)\left[\frac{3}{4}\left(1 + \sum_{i=1}^{n-1}\lambda_i^2\right) - \frac{1}{2}\left(1 + \sum_{i=1}^{n-1}\lambda_i^3\right) - \frac{n(n+1)}{4(n-1)^2}\right] &\geq 0,
\end{aligned}$$

where the inequality comes from the preceding lemma. □